

Shorter proofs of some recent even-tupled coincidence theorems for weak contractions in ordered metric spaces

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Abstract In this paper, we prove some recent even coincidence theorems due to Imdad et al. (Bull Math Anal Appl 5(4): 19-39, 2013) using a method of reduction from the respective coincidence theorems for mappings with one variable in ordered complete metric spaces. Our technique of proof is different, slightly simpler, shorter and more effective than the ones used in Imdad et al.

Keywords Partially ordered set · Compatible mapping · Mixed g -monotone property · n -tupled coincidence point · n -tupled fixed point

Mathematics Subject Classification 54H10 · 54H25

Introduction

The investigation of fixed points in ordered metric spaces is a relatively new development which appears to have its origin in the paper of Ran and Reurings [30] which was well complimented by Nieto and López [25]. Ran and Reurings' fixed point theorem extended and refined by many authors, (for details see [8, 12, 24–27, 37]).

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [11]. In [5], Bhaskar and La-

kshmikantham introduced the notion of mixed monotone property for a mapping $F : X^2 \rightarrow X$ and proved some coupled fixed point theorems for weakly linear contractions enjoying mixed monotone property in ordered complete metric spaces. In this continuation, Lakshmikantham and Ćirić [22] generalized these results for nonlinear contraction mappings by introducing two ideas namely: coupled coincidence point and mixed g -monotone property. In an attempt to extend the definition from X^2 to X^3 , Berinde and Borcut [4] introduced the concept of tripled fixed point and utilize the same to prove some tripled fixed point theorems. After that, Karapinar [16] introduced the quadrupled fixed point to prove some quadrupled fixed point theorems for nonlinear contraction mappings satisfying mixed g -monotone property (for more details see [17, 18]). Recently, Samet and Vetro [32] extended the idea of coupled as well as quadrupled fixed point to higher dimensions by introducing the notion of fixed point of n -order (or n -tupled fixed point, where $n \in \mathbb{N}$ and $n \geq 3$) and presented some n -tupled fixed point results in complete metric spaces, using a new concept of f -invariant set. Here it can be pointed out that the notion of tripled fixed point due to Berinde and Borcut [4] is different from the one defined by Samet and Vetro [32] for $n = 3$ in the case of ordered metric spaces in order to keep the mixed monotone property working. Recently, Imdad et al. [13] extended the idea of mixed g -monotone property to the mapping $F : X^n \rightarrow X$ (where n is even natural number) and proved an even-tupled coincidence point theorem for nonlinear contraction mappings satisfying mixed g -monotone property. Basically their results are true for only even n but not for odd ones (for details see [15]). Further, Imdad et al. [14] proved some even-tupled coincidence theorems under nonlinear weak contractions due to Choudhury et al. [9].

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Very recently, Samet et al. [34] have shown that the coupled (analogously n -tupled) fixed results can be more easily obtained using well-known fixed point theorems on ordered metric spaces (see also [10, 28, 29]). This technique of proof is different, slightly simpler, shorter and more effective than classical technique. In this paper, we prove the main results of Imdad et al. [14] following the techniques of Samet et al. [34].

Preliminaries

With a view to make, our presentation self-contained, we collect some basic definitions and needed results which will be used frequently in the text later.

Definition 2.1 Let X be a non-empty set. A relation ' \preceq ' on X is said to be a partial order if the following properties are satisfied:

- (i) reflexive: $x \preceq x$ for all $x \in X$,
- (ii) anti-symmetric: $x \preceq y$ and $y \preceq x$ imply $x = y$,
- (iii) transitive: $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for all $x, y, z \in X$.

A non-empty set X together with a partial order ' \preceq ' is said to be an ordered set and we denote it by (X, \preceq) .

Definition 2.2 Let (X, \preceq) be an ordered set. Any two elements x and y are said to be comparable elements in X if either $x \preceq y$ or $y \preceq x$.

Definition 2.3 ([27]) A triplet (X, d, \preceq) is called an ordered metric space if (X, d) is a metric space and (X, \preceq) is an ordered set. Moreover, if d is a complete metric on X , then we say that (X, d, \preceq) is an ordered complete metric space.

Recently, Kutbi et al. [21] introduced the concept of regular map.

Definition 2.4 ([21]) An ordered metric space (X, d, \preceq) is said to be nondecreasing regular (resp. nonincreasing regular) if it satisfies the following property: if $\{x_m\}$ is a nondecreasing (resp. nonincreasing) sequence and $x_m \rightarrow x$, then $x_m \preceq x$ (resp. $x \preceq x_m$) $\forall m \in \mathbb{N} \cup \{0\}$.

Definition 2.5 ([21]) An ordered metric space (X, d, \preceq) is said to be regular if it is both nondecreasing regular and nonincreasing regular.

Definition 2.6 Let (X, d, \preceq) be an ordered metric space and $g : X \rightarrow X$ be a mapping. Then X is said to be nondecreasing g -regular (resp. nonincreasing g -regular) if it satisfies the following property: if $\{x_m\}$ is a nondecreasing (resp. nonincreasing) sequence and $x_m \rightarrow x$, then $gx_m \preceq gx$ (resp. $gx \preceq gx_m$) $\forall m \in \mathbb{N} \cup \{0\}$.

Definition 2.7 An ordered metric space (X, d, \preceq) is said to be g -regular if it is both nondecreasing g -regular and nonincreasing g -regular.

Notice that, on setting $g = I$ (identity mapping on X), Definitions 2.6 and 2.7 reduce to Definitions 2.4 and 2.5 respectively.

Throughout the paper, n stands for a general even natural number. Let us denote by X^n the product space $X \times X \times \dots \times X$ of n identical copies of X .

Definition 2.8 ([13]) Let (X, \preceq) be an ordered set and $F : X^n \rightarrow X$ and $g : X \rightarrow X$ be two mappings. Then F is said to have the mixed g -monotone property if F is g -nondecreasing in its odd position arguments and g -nonincreasing in its even position arguments, that is, for $x^1, x^2, x^3, \dots, x^n \in X$,

$$\begin{aligned} & \text{if} \\ & \text{for all } x_1^1, x_2^1 \in X, \quad gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, \dots, x^n) \\ & \preceq F(x_2^1, x^2, x^3, \dots, x^n) \\ & \text{for all } x_1^2, x_2^2 \in X, \quad gx_1^2 \preceq gx_2^2 \Rightarrow F(x^1, x_1^2, x^3, \dots, x^n) \preceq \\ & F(x^1, x_2^2, x^3, \dots, x^n) \\ & \text{for all } x_1^3, x_2^3 \in X, \quad gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, \dots, x^n) \preceq \\ & F(x^1, x^2, x_2^3, \dots, x^n) \\ & \vdots \\ & \text{for all } x_1^n, x_2^n \in X, \quad gx_1^n \preceq gx_2^n \Rightarrow F(x^1, x^2, x^3, \dots, x_1^n) \preceq \\ & F(x^1, x^2, x^3, \dots, x_2^n). \end{aligned}$$

For $g = I$ (identity mapping), Definition 2.8 reduces to mixed monotone property (for details see [13]).

Definition 2.9 ([32]) An element $(x^1, x^2, \dots, x^n) \in X^n$ is called an n -tupled fixed point of the mapping $F : X^n \rightarrow X$ if

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = x^1 \\ F(x^2, x^3, \dots, x^n, x^1) = x^2 \\ F(x^3, \dots, x^n, x^1, x^2) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = x^n. \end{cases}$$

Definition 2.10 ([13]) An element $(x^1, x^2, \dots, x^n) \in X^n$ is called an n -tupled coincidence point of mappings $F : X^n \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = g(x^1) \\ F(x^2, x^3, \dots, x^n, x^1) = g(x^2) \\ F(x^3, \dots, x^n, x^1, x^2) = g(x^3) \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = g(x^n). \end{cases}$$

Remark 2.1 For $n = 2$, Definitions 2.9 and 2.10 yield the definitions of coupled fixed point and coupled coincidence point respectively while on the other hand, for $n = 4$ these

definitions yield the definitions of quadrupled fixed point and quadrupled coincidence point respectively.

Definition 2.11 An element $(x^1, x^2, \dots, x^n) \in X^n$ is called an n -tupled common fixed point of mappings $F : X^n \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = g(x^1) = x^1 \\ F(x^2, x^3, \dots, x^n, x^1) = g(x^2) = x^2 \\ F(x^3, \dots, x^n, x^1, x^2) = g(x^3) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = g(x^n) = x^n. \end{cases}$$

Definition 2.12 ([14]) Let X be a non-empty set. Then the mappings $F : X^n \rightarrow X$ and $g : X \rightarrow X$ are said to be compatible if

$$\begin{cases} \lim_{m \rightarrow \infty} d(g(F(x_m^1, x_m^2, \dots, x_m^n)), F(gx_m^1, gx_m^2, \dots, gx_m^n)) = 0 \\ \lim_{m \rightarrow \infty} d(g(F(x_m^2, \dots, x_m^n, x_m^1)), F(gx_m^2, \dots, gx_m^n, x_m^1)) = 0 \\ \vdots \\ \lim_{m \rightarrow \infty} d(g(F(x_m^n, x_m^1, \dots, x_m^{n-1})), F(gx_m^n, gx_m^1, \dots, gx_m^{n-1})) = 0, \end{cases}$$

where $\{x_m^1\}, \{x_m^2\}, \dots, \{x_m^n\}$ are sequences in X such that

$$\begin{cases} \lim_{m \rightarrow \infty} F(x_m^1, x_m^2, \dots, x_m^n) = \lim_{m \rightarrow \infty} g(x_m^1) = x^1 \\ \lim_{m \rightarrow \infty} F(x_m^2, \dots, x_m^n, x_m^1) = \lim_{m \rightarrow \infty} g(x_m^2) = x^2 \\ \vdots \\ \lim_{m \rightarrow \infty} F(x_m^n, x_m^1, \dots, x_m^{n-1}) = \lim_{m \rightarrow \infty} g(x_m^n) = x^n, \end{cases}$$

for some $x^1, x^2, \dots, x^n \in X$ are satisfied.

The following families of control functions are indicated in Choudhury et al. [9].

$\mathfrak{S} = \{\zeta : [0, \infty) \rightarrow [0, \infty) \mid \zeta \text{ is continuous and } \zeta(t) = 0 \text{ if and only if } t = 0\}$

$\Omega := \{\varphi : [0, \infty) \rightarrow [0, \infty) : \varphi \text{ is continuous and monotone nondecreasing and } \varphi(t) = 0 \text{ if and only if } t = 0\}$

Notice that members of Ω are called altering distance functions (cf. [20]).

Now, we state the main result of Imdad et al. [14], which is indeed n -tupled extension of that of Choudhury et al. [9].

Theorem 2.1 Let (X, d, \preceq) be an ordered complete metric space and $F : X^n \rightarrow X$ and $g : X \rightarrow X$ be two mappings. Suppose that the following conditions are satisfied:

- (i) $F(X^n) \subseteq g(X)$,
- (ii) F and g are compatible,

- (iii) F has the mixed g -monotone property,
- (iv) g is continuous,
- (v) either F is continuous or X is g -regular,
- (vi) there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ such that

$$\begin{cases} gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n) \\ F(x_0^2, x_0^3, \dots, x_0^n, x_0^1) \preceq gx_0^2 \\ gx_0^3 \preceq F(x_0^3, \dots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}) \preceq gx_0^n, \end{cases}$$

- (vii) there exist $\varphi \in \Omega$ and $\zeta \in \mathfrak{S}$ such that

$$\begin{aligned} \varphi(d(FU, FV)) &\leq \varphi\left(\max_{1 \leq i \leq n} d(gx^i, gy^i)\right) \\ &\quad - \zeta\left(\max_{1 \leq i \leq n} d(gx^i, gy^i)\right), \end{aligned}$$

for all $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n) \in X^n$ with $gy^1 \preceq gx^1, gx^2 \preceq gy^2, gy^3 \preceq gx^3, \dots, gx^n \preceq gy^n$. Then F and g have an n -tupled coincidence point.

Main results

Let (X, \preceq) be an ordered set. Define the following partial order \sqsubseteq on the product space X^n , for $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n) \in X^n$

$$U \sqsubseteq V \Leftrightarrow x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \dots, y^n \preceq x^n.$$

Let (X, d) be a metric space. Define the following metric \tilde{D} on the product space X^n , for $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n) \in X^n$,

$$\tilde{D}(U, V) = \max_{1 \leq i \leq n} d(x^i, y^i).$$

The proofs of the following lemmas follow immediately. We note the same idea here, but in the case of coupled and tripled fixed point theorems, we have been first used in ([3, 28, 33]).

Lemma 3.1 Let (X, d, \preceq) be an ordered complete metric space. Then $(X^n, \tilde{D}, \sqsubseteq)$ is an ordered complete metric space.

Lemma 3.2 Let (X, d, \preceq) be an ordered metric space and $F : X^n \rightarrow X$ and $g : X \rightarrow X$ be two mappings. Define mappings $T_F : X^n \rightarrow X^n$ and $T_g : X^n \rightarrow X^n$ by

$$T_F(x^1, x^2, \dots, x^n) = (F(x^1, x^2, \dots, x^n), F(x^2, \dots, x^n, x^1), \dots, F(x^n, x^1, \dots, x^{n-1}))$$

and $T_g(x^1, x^2, \dots, x^n) = (gx^1, gx^2, \dots, gx^n)$. Then the following hold:



- (1) If F has the mixed g -monotone property, then T_F is monotone T_g -nondecreasing with respect to \sqsubseteq .
- (2) If F and g are compatible, then T_F and T_g are compatible.
- (3) If g is continuous, then T_g is continuous.
- (4) If F is continuous, then T_F is continuous.
- (5) If (X, d, \preceq) is g -regular, then $(X^n, \tilde{D}, \sqsubseteq)$ is nondecreasing g -regular.
- (6) A point $(x^1, x^2, \dots, x^n) \in X^n$ is an n -tupled coincidence point of F and g iff (x^1, x^2, \dots, x^n) is a coincidence point of T_F and T_g .

The following lemma is crucial for our main result.

Lemma 3.3 Let (X, d, \preceq) be an ordered complete metric space and f and g be two self-mappings on X . Suppose that the following conditions are satisfied:

- (i) $f(X) \subseteq g(X)$,
- (ii) f is monotone g -nondecreasing,
- (iii) f and g are compatible,
- (iv) g is continuous,
- (v) either f is continuous or X is nondecreasing g -regular,
- (vi) there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$,
- (vii) there exist $\varphi \in \Omega$ and $\zeta \in \mathfrak{I}$ such that for all $x, y \in X$,

$$\varphi(d(f(x), f(y))) \leq \varphi(d(g(x), g(y))) - \zeta(d(g(x), g(y))), \text{ with } g(x) \preceq g(y). \quad (3.1)$$

Then f and g have a coincidence point.

Proof In view of assumption (vi), if $g(x_0) = f(x_0)$, then x_0 is a coincidence point of f and g and hence proof is finished. On the other hand if $g(x_0) \neq f(x_0)$, then we have $g(x_0) \prec f(x_0)$. So according to assumption (i), that is, $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $g(x_1) = f(x_0)$. Again from $f(X) \subseteq g(X)$, we can choose $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process, we define a sequence $\{x_m\} \subset X$ of joint iterates such that

$$g(x_{m+1}) = f(x_m) \quad \forall m \in \mathbb{N} \cup \{0\}. \quad (3.2)$$

Now, we assert that $\{g(x_m)\}$ is a nondecreasing sequence, that is

$$g(x_m) \preceq g(x_{m+1}) \quad \forall m \in \mathbb{N} \cup \{0\}. \quad (3.3)$$

We prove this fact by mathematical induction. On using (3.2) for $m = 0$ and assumption (vi), we have

$$g(x_0) \preceq f(x_0) = g(x_1).$$

Thus, (3.3) holds for $m = 0$. Suppose that (3.3) holds for $m = r > 0$, that is,

$$g(x_r) \preceq g(x_{r+1}). \quad (3.4)$$

Then we have to show that (3.3) holds for $m = r + 1$. To accomplish this we use (3.2), (3.4) and assumption (ii) so that

$$g(x_{r+1}) = f(x_r) \preceq f(x_{r+1}) = g(x_{r+2}).$$

Thus, by induction, (3.3) holds for all $m \in \mathbb{N} \cup \{0\}$.

If $g(x_m) = g(x_{m+1})$ for some $m \in \mathbb{N}$, then using (3.2), we have $g(x_m) = f(x_m)$, that is, x_m is a coincidence point of f and g and hence proof is finished. On the other hand if $g(x_m) \neq g(x_{m+1})$ for each $m \in \mathbb{N} \cup \{0\}$, we can define a sequence

$$\delta_m := d(g(x_m), g(x_{m+1})), \quad m \in \mathbb{N} \cup \{0\}. \quad (3.5)$$

On using (3.2), (3.3), (3.5) and assumption (vii), we obtain

$$\begin{aligned} \varphi(\delta_{m+1}) &= \varphi(d(g(x_{m+1}), g(x_{m+2}))) \\ &= \varphi(d(f(x_m), f(x_{m+1}))) \\ &\leq \varphi(d(g(x_m), g(x_{m+1}))) - \zeta(d(g(x_m), g(x_{m+1}))) \\ &= \varphi(\delta_m) - \zeta(\delta_m). \end{aligned} \quad (3.6)$$

On using the property of φ , we have $\varphi(\delta_{m+1}) \leq \varphi(\delta_m)$, which implies that $\delta_{m+1} \leq \delta_m$. Therefore, $\{\delta_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $\delta \geq 0$ such that $\delta_m \rightarrow \delta$ as $m \rightarrow \infty$. Taking limit as $m \rightarrow \infty$ in (3.6) and using the continuities of φ and ζ , we have $\varphi(\delta) \leq \varphi(\delta) - \zeta(\delta)$, which is a contradiction under $r = 0$. Therefore,

$$\lim_{m \rightarrow \infty} \delta_m = \lim_{m \rightarrow \infty} d(g(x_m), g(x_{m+1})) = 0. \quad (3.7)$$

Now, we show that $\{g(x_m)\}$ is a Cauchy sequence. On the contrary, suppose that $\{g(x_m)\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{t(k)\}$ such that for all positive integers k , $t(k) > m(k) > k$, such that

$$\eta_k = d(g(x_{m(k)}), g(x_{t(k)})) \geq \epsilon, \text{ and } d(g(x_{m(k)}), g(x_{t(k)-1})) < \epsilon.$$

Now,

$$\begin{aligned} \epsilon \leq \eta_k &= d(g(x_{m(k)}), g(x_{t(k)})) \\ &\leq d(g(x_{m(k)}), g(x_{t(k)-1})) + d(g(x_{t(k)-1}), g(x_{t(k)})) \\ &< \epsilon + \delta_{t(k)-1} \end{aligned}$$

that is,

$$\epsilon \leq \eta_k < \epsilon + \delta_{t(k)-1}.$$

Letting $k \rightarrow \infty$ in above inequality and using (3.7), we get

$$\lim_{k \rightarrow \infty} \eta_k = \epsilon. \quad (3.8)$$

Again,

$$\begin{aligned}
\eta_{k+1} &= d(g(x_{m(k)+1}), g(x_{t(k)+1})) \\
&\leq d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(x_{m(k)}), g(x_{t(k)})) \\
&\quad + d(g(x_{t(k)}), g(x_{t(k)+1})) \\
&< \delta_{m(k)+1} + \eta_k + \delta_{t(k)+1} \\
&\Rightarrow \eta_{k+1} < \delta_{m(k)+1} + \eta_k + \delta_{t(k)+1}.
\end{aligned}$$

Letting $k \rightarrow \infty$ in above inequality and using (3.7) and (3.8), we get

$$\lim_{k \rightarrow \infty} \eta_{k+1} = \epsilon. \quad (3.9)$$

Since $t(k) > m(k)$, hence by (3.3), we get $g(x_{m(k)}) \leq g(x_{t(k)})$. Therefore, owing to (3.1) and assumption (vii), we get

$$\begin{aligned}
\varphi(\eta_{k+1}) &= \varphi(d(g(x_{m(k)+1}), g(x_{t(k)+1}))) \\
&= \varphi(d(f(x_{m(k)}), f(x_{t(k)}))) \\
&\leq \varphi(d(g(x_{m(k)}), g(x_{t(k)}))) - \zeta(d(g(x_{m(k)}), g(x_{t(k)}))) \\
&= \varphi(\eta_k) - \zeta(\eta_k)
\end{aligned}$$

that is,

$$\varphi(\eta_{k+1}) \leq \varphi(\eta_k) - \zeta(\eta_k).$$

Letting $k \rightarrow \infty$ in above inequality and using (3.8), (3.9) and continuities of φ and ζ , we get

$$\varphi(\epsilon) \leq \varphi(\epsilon) - \zeta(\epsilon)$$

which is a contradiction by virtue of property of ζ . Therefore, the sequence $\{g(x_m)\}$ is Cauchy. From the completeness of X , there exists $x \in X$ such that

$$\lim_{m \rightarrow \infty} f(x_m) = \lim_{m \rightarrow \infty} g(x_m) = x. \quad (3.10)$$

Since F and g are compatible, we have from (3.10),

$$\lim_{m \rightarrow \infty} d(f(gx_m), g(fx_m)) = 0. \quad (3.11)$$

Now, we use assumption (v). Firstly, we assume that f is continuous. Then for all $m \in \mathbb{N} \cup \{0\}$, we have

$$d(g(x), f(gx_m)) \leq d(g(x), g(fx_m)) + d(g(fx_m), f(gx_m)).$$

Taking $k \rightarrow \infty$ in above inequality and using (3.10), (3.11) and continuities of f and g , we get $d(g(x), f(x)) = 0$, that is, $g(x) = f(x)$. Hence, the element $x \in X$ is a coincidence point of f and g . Next, we suppose that X is nondecreasing g -regular. From (3.3) and (3.10), we get

$$g(gx_m) \preceq g(x). \quad (3.12)$$

Since f and g are compatible and g is continuous by (3.10) and (3.11), we have

$$\lim_{m \rightarrow \infty} g(gx_m) = g(x) = \lim_{m \rightarrow \infty} g(fx_m) = \lim_{m \rightarrow \infty} f(gx_m). \quad (3.13)$$

Now, using triangle inequality, we have

$$\begin{aligned}
d(f(x), g(x)) &\leq d(f(x), g(gx_{m+1})) + d(g(gx_{m+1}), g(x)) \\
&= d(f(x), g(fx_m)) + d(g(gx_{m+1}), g(x)).
\end{aligned}$$

Taking $k \rightarrow \infty$ in above inequality and using (3.13), we have

$$\begin{aligned}
d(f(x), g(x)) &\leq \lim_{m \rightarrow \infty} d(f(x), g(fx_m)) + \lim_{m \rightarrow \infty} d(g(gx_{m+1}), g(x)) \\
&= \lim_{m \rightarrow \infty} d(f(x), f(gx_m)).
\end{aligned}$$

Since φ is continuous and monotone nondecreasing, from the above inequality we have

$$\begin{aligned}
\varphi(d(f(x), g(x))) &\leq \varphi(\lim_{m \rightarrow \infty} d(f(x), f(gx_m))) \\
&= \lim_{m \rightarrow \infty} \varphi(d(f(x), f(gx_m))).
\end{aligned}$$

By (3.12) and assumption (vii), we get

$$\begin{aligned}
\varphi(d(f(x), g(x))) &\leq \lim_{m \rightarrow \infty} \varphi(d(f(x), f(gx_m))) \\
&= \lim_{m \rightarrow \infty} \varphi(d(g(x), g(gx_m))) \\
&\quad + \lim_{m \rightarrow \infty} \zeta(d(g(x), g(gx_m))).
\end{aligned}$$

Using (3.13) and the properties of φ and ζ , we have $\varphi(d(f(x), g(x))) = 0$, which implies that $d(f(x), g(x)) = 0$, that is, $g(x) = f(x)$. Hence, $x \in X$ is a coincidence point of f and g .

Lemma 3.4 In addition to the hypotheses of Lemma 3.3, suppose that for real $x, y \in X$ there exists, $z \in X$ such that $f(z)$ is comparable to $f(x)$ and $f(y)$. Then f and g have a unique common fixed point.

Proof The set of coincidence points of f and g is non-empty due to Lemma 3.3. Assume now, x and y are two coincidence points of f and g , that is,

$$f(x) = g(x) \text{ and } f(y) = g(y).$$

Now we will show that $g(x) = g(y)$. By assumption, there exists $z \in X$ such that $f(z)$ is comparable to $f(x)$ and $f(y)$. Put $z_0 = z$ and choose $z_1 \in X$ such that $g(z_1) = f(z_0)$. Further define sequence $\{g(z_m)\}$ such that $g(z_{m+1}) = f(z_m)$. Further set $x_0 = x$ and $y_0 = y$. In the same way, define the sequences $\{g(x_m)\}$ and $\{g(y_m)\}$. Then, it is easy to show that

$$g(x_{m+1}) = f(x_m) \text{ and } g(y_{m+1}) = f(y_m).$$

Since $f(x) = g(x_1) = g(x)$ and $f(z) = g(z_1)$ are comparable, we have

$$g(x) \preceq g(z_1).$$

It is easy to show that $g(x)$ and $g(z_m)$ are comparable, that is, for all $m \in \mathbb{N}$,



$$g(x) \preceq g(z_m).$$

Thus from (3.1) we have

$$\begin{aligned} \varphi(d(g(x), g(z_{m+1}))) &= \varphi(d(f(x), f(z_m))) \\ &\leq \varphi(d(g(x), g(z_m))) - \zeta(d(g(x), g(z_m))). \end{aligned}$$

Let $R_m = d(g(x), g(z_{m+1}))$. Then

$$\varphi(R_m) \leq \varphi(R_{m-1}) - \zeta(R_{m-1}). \quad (3.14)$$

Using the property of φ , we have $\varphi(R_m) \leq \varphi(R_{m-1})$, which implies that $R_m \leq R_{m-1}$ (by the property of φ). Therefore $\{R_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence, there exists $r \geq 0$ such that $R_m \rightarrow r$ as $m \rightarrow \infty$. Taking the limit as $m \rightarrow \infty$ in (3.14) and using the continuities of φ and ζ , we have $\varphi(r) \leq \varphi(r) - \zeta(r)$, which is a contradiction unless $r = 0$. Therefore $R_m \rightarrow 0$ as $m \rightarrow \infty$, that is,

$$\lim_{m \rightarrow \infty} d(g(x), g(z_{m+1})) = 0.$$

Similarly we can prove that

$$\lim_{m \rightarrow \infty} d(g(y), g(z_{m+1})) = 0.$$

Therefore by triangle inequality

$$\begin{aligned} d(g(x), g(y)) &\leq d(g(x), g(z_{m+1})) + d(g(z_{m+1}), \\ &g(y)) \rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Hence

$$g(x) = g(y). \quad (3.15)$$

Since $g(x) = f(x)$ and f and g are compatible, we have $gg(x) = f(gx)$. Write $g(x) = a$, then we have

$$g(a) = f(a). \quad (3.16)$$

Thus a is the coincidence point of f and g . Then owing to (3.15) with $y = a$, it follows that $g(x) = g(a)$, that is,

$$g(a) = a. \quad (3.17)$$

Using (3.16) and (3.17), we have $a = g(a) = f(a)$. Thus a is the common fixed point of f and g . To prove the uniqueness, assume that b is another common fixed point of f and g . Then by (3.15), we have

$$b = g(b) = g(a) = a.$$

This completes the proof of Lemma.

Theorem 3.1 *Theorem 2.1 is obtained using Lemmas 3.1, 3.2 and 3.3.*

Proof Consider the product space $Y = X^n$ equipped with the metric \tilde{D} [given by (B)] and the partial order \sqsubseteq [given by (A)]. Then by Lemma 3.1, $(Y, \tilde{D}, \sqsubseteq)$ is an ordered

complete metric space. Also F and g induce mappings $T_F : Y \rightarrow Y$ and $T_g : Y \rightarrow Y$ (defined in Lemma 3.2). Clearly,

- (i) implies that $T_F(Y) \subseteq T_g(Y)$,
- (ii) implies that T_F is monotone T_g -nondecreasing (by item (1) of Lemma 3.2),
- (iii) implies that T_F and T_g are compatible (by item (2) of Lemma 3.2),
- (iv) implies that T_g is continuous (by item (3) of Lemma 3.2),
- (v) implies that either T_F is continuous [by item (4) of Lemma 3.2] or $(Y, \tilde{D}, \sqsubseteq)$ is nondecreasing g -regular [by item (5) of Lemma 3.2],
- (vi) is equivalent to the condition: there exists $U_0 = (x_0^1, x_0^2, \dots, x_0^n) \in Y$ such that $T_g(U_0) \subseteq T_F(U_0)$.

Now, in view of (vii), for given $U, V \in Y$ such that $T_g(U) \sqsubseteq T_g(V)$ implies that

$$(gx^1, gx^2, \dots, gx^n) \sqsubseteq (gy^1, gy^2, \dots, gy^n).$$

It follows that for odd i ,

$$\begin{aligned} (gx^i, gx^{i+1}, \dots, gx^n, gx^1, gx^2, \dots, gx^{i-1}) &\sqsubseteq (gy^i, gy^{i+1}, \dots, gy^n, \\ &gy^1, gy^2, \dots, gy^{i-1}), \end{aligned} \quad (3.18)$$

and for even i ,

$$\begin{aligned} (gy^i, gy^{i+1}, \dots, gy^n, gy^1, gy^2, \dots, gy^{i-1}) &\sqsubseteq (gx^i, gx^{i+1}, \\ &\dots, gx^n, gx^1, gx^2, \dots, gx^{i-1}). \end{aligned} \quad (3.19)$$

If i is odd, then using (3.18) and (vii), we get

$$\begin{aligned} &d(F(x^i, x^{i+1}, \dots, x^n, x^1, x^2, \dots, x^{i-1}), F(y^i, y^{i+1}, \dots, y^n, y^1, y^2, \dots, y^{i-1})) \\ &\leq \varphi(\max\{d(gx^i, gy^i), d(gx^{i+1}, gy^{i+1}), \dots, d(gx^n, gy^n), d(gx^1, gy^1), \\ &d(gx^2, gy^2), \dots, d(gx^{i-1}, gy^{i-1})\}) - \zeta(\max\{d(gx^i, gy^i), d(gx^{i+1}, gy^{i+1}), \dots, \\ &d(gx^n, gy^n), d(gx^1, gy^1), d(gx^2, gy^2), \dots, d(gx^{i-1}, gy^{i-1})\}) \\ &= \varphi(\max_{1 \leq i \leq n} d(gx^i, gy^i)) - \zeta(\max_{1 \leq i \leq n} d(gx^i, gy^i)). \end{aligned}$$

If i is even, then using (3.19) and (vii), we get

$$\begin{aligned} &d(F(x^i, x^{i+1}, \dots, x^n, x^1, x^2, \dots, x^{i-1}), F(y^i, y^{i+1}, \dots, y^n, y^1, y^2, \dots, y^{i-1})) \\ &= d(F(y^i, y^{i+1}, \dots, y^n, y^1, y^2, \dots, y^{i-1}), F(x^i, x^{i+1}, \dots, x^n, x^1, x^2, \dots, x^{i-1})) \\ &\leq \varphi(\max\{d(gy^i, gx^i), d(gy^{i+1}, gx^{i+1}), \dots, d(gy^n, gx^n), d(gy^1, gx^1), \\ &d(gy^2, gx^2), \dots, d(gy^{i-1}, gx^{i-1})\}) - \zeta(\max\{d(gy^i, gx^i), d(gy^{i+1}, gx^{i+1}), \dots, \\ &d(gy^n, gx^n), d(gy^1, gx^1), d(gy^2, gx^2), \dots, d(gy^{i-1}, gx^{i-1})\}) \\ &= \varphi(\max_{1 \leq i \leq n} d(gx^i, gy^i)) - \zeta(\max_{1 \leq i \leq n} d(gx^i, gy^i)). \end{aligned}$$

Hence, in both the cases, for each i ($1 \leq i \leq n$), we have

$$\begin{aligned} &d(F(x^i, x^{i+1}, \dots, x^n, x^1, x^2, \dots, x^{i-1}), F(y^i, y^{i+1}, \dots, y^n, \\ &y^1, y^2, \dots, y^{i-1})) \\ &\leq \varphi(\max_{1 \leq i \leq n} d(gx^i, gy^i)) - \zeta(\max_{1 \leq i \leq n} d(gx^i, gy^i)). \end{aligned} \quad (3.20)$$



Hence using (3.20), we have

$$\begin{aligned} & \tilde{D}(T_F(U), T_F(V)) \\ &= \max_{1 \leq i \leq n} d(F(x^i, x^{i+1}, \dots, x^n, x^1, x^2, \dots, x^{i-1}), \\ & \quad F(y^i, y^{i+1}, \dots, y^n, y^1, y^2, \dots, y^{i-1})) \\ &\leq \max_{1 \leq i \leq n} [\varphi(\max_{1 \leq i \leq n} d(gx^i, gy^i)) - \zeta(\max_{1 \leq i \leq n} d(gx^i, gy^i))] \\ &= \varphi(\max_{1 \leq i \leq n} d(gx^i, gy^i)) - \zeta(\max_{1 \leq i \leq n} d(gx^i, gy^i)) \\ &= \varphi(\tilde{D}(T_g(U), T_g(V))) - \zeta(\tilde{D}(T_g(U), T_g(V))). \end{aligned}$$

Thus all conditions of Lemma 3.3 are satisfied for ordered complete metric space $(Y, \tilde{D}, \sqsubseteq)$ and mappings $T_F : Y \rightarrow Y$ and $T_g : Y \rightarrow Y$. Therefore, T_F and T_g have a coincidence point in $Y = X^n$. According to item (6) of Lemma 3.2, the mappings F and g have an n -tupled coincidence point.

Corollary 3.1 *Let (X, d, \preceq) be an ordered complete metric space and $F : X^n \rightarrow X$ be a mapping. Suppose that the following conditions are satisfied:*

- (i) F has the mixed monotone property,
- (ii) either F is continuous or X is regular,
- (iii) there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n) \\ F(x_0^2, x_0^3, \dots, x_0^n, x_0^1) \preceq x_0^2 \\ x_0^3 \preceq F(x_0^3, \dots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}) \preceq x_0^n, \end{cases}$$

- (iv) there exist $\varphi \in \Omega$ and $\zeta \in \mathfrak{F}$ such that

$$\varphi(d(FU, FV)) \leq \varphi(\max_{1 \leq i \leq n} d(x^i, y^i)) - \zeta(\max_{1 \leq i \leq n} d(x^i, y^i)),$$

for all $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \dots, y^n \preceq x^n$.

Then F has an n -tupled fixed point.

Proof It is sufficient to take $g = I$ (identity mapping) in Theorem 3.1. \square

Corollary 3.2 *Corollary 3.1 remains true if condition (iv) is replaced by the following: (iv)' there exists $\zeta \in \mathfrak{F}$ such that*

$$d(FU, FV) \leq \max_{1 \leq i \leq n} d(x^i, y^i) - \zeta(\max_{1 \leq i \leq n} d(x^i, y^i)),$$

for all $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \dots, y^n \preceq x^n$.

Proof It is sufficient to take φ and g to be identity mappings in Theorem 3.1.

Corollary 3.3 *Corollary 3.1 remains true if condition (iv) is replaced by the following:*

(iv)'' there exists $k \in (0, 1)$ such that

$$d(FU, FV) \leq k \max_{1 \leq i \leq n} d(x^i, y^i),$$

for all $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \dots, y^n \preceq x^n$.

Proof It is sufficient to take φ and g to be identity mappings and $\zeta(t) = (1 - k)t$, $k \in (0, 1)$ in Theorem 3.1.

Remark 3.1

- On setting $n = 2$ in Theorem 3.1, we get Theorem 3.1 of Choudhury et al. [9].
- On setting $n = 2$ in Corollaries 3.1–3.3, we get Corollaries 3.2–3.4 of Choudhury et al. [9].
- On setting $n = 4$ in Theorem 3.1 and Corollaries 3.1–3.3, we get their corresponding quadrupled fixed point results.

Now we shall prove the uniqueness of n -tupled fixed point.

Theorem 3.2 *In addition to the hypotheses of Theorem 3.1, suppose that for real (x^1, x^2, \dots, x^n) and $(y^1, y^2, \dots, y^n) \in X^n$ there exists, $(z^1, z^2, \dots, z^n) \in X^n$ such that $(F(z^1, z^2, \dots, z^n), F(z^2, \dots, z^n, z^1), \dots, F(z^n, z^1, \dots, z^{n-1}))$ is comparable to $(F(x^1, x^2, \dots, x^n), F(x^2, \dots, x^n, x^1), \dots, F(x^n, x^1, \dots, x^{n-1}))$ and $(F(y^1, y^2, \dots, y^n), F(y^2, \dots, y^n, y^1), \dots, F(y^n, y^1, \dots, y^{n-1}))$. Then F and g have a unique n -tupled common fixed point.*

Proof Set $U = (x^1, x^2, \dots, x^n)$, $V = (y^1, y^2, \dots, y^n)$ and $W = (z^1, z^2, \dots, z^n)$. Then we have

$$T_F(W) \sqsubseteq T_F(U) \text{ or } T_F(U) \sqsubseteq T_F(W)$$

and

$$T_F(W) \sqsubseteq T_F(V) \text{ or } T_F(V) \sqsubseteq T_F(W).$$

Hence using Lemma 3.4, T_F and T_g have a unique n -tupled common fixed point. \square

Remark 3.2 From Theorem 3.2, for $n = 2$, we can get unique coupled common fixed point theorem contained in Choudhury et al. [9].

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